Basic Strategy for Card-Counters: An Analytic Approach

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Abstract
We develop an efficient method, similar to that of Marcus (2007) who calls it Counter Basic Strategy, for casino blackjack card counters to play the cards of each hand. It is as easy to use as Basic Strategy, unlike elaborate schemes involving strategy indices; and captures much of the yield improvement they offer. This method is taken to be that set of count- and composition-independent play parameters (like Basic Strategy in this regard) that maximizes the yield, the expected cash flow averaged over all rounds between shuffles. Basic Strategy, in contrast, maximizes the expected return of just the first round after a shuffle.

1. Introduction

Most players who use a card counting technique do so in order to gauge the appropriate amount to bet on the next hand. But it is well known that the ideal way to play that hand depends on the un-dealt cards, for which counting also gives a reasonable indicator. Yet this 'ideal' way is quite complex, involving many changes of the decision parameters as the count varies through its wide range of statistically likely values. The parameters involved - altogether more than two dozen in number - are: when to stand vs. draw (for hard and soft hands separately); which hands (hard and soft) to double down; which pairs to split; and which hands to insure or surrender. Because the 'ideal' is generally regarded as difficult to carry out in a casino – approaching, some might say, the super-human – various simplifications have been proposed; one or another is used by almost all card counters.

These simplifications can be classified by how much information the player uses to make his playing decisions. The 'ideal' is one in which he employs the individual identities of every card he is dealt, along with that of the dealer's upcard. With only slightly reduced performance, the player uses the identities of just the first two cards in his hand, plus the upcard; the decision rules for subsequent cards are fixed. Such a class is usually referred to as "composition-dependent". Easier to manage in practice is the class ("total-dependent") using just the total value of the two initial cards, not their identities (although still recognizing opportunities to double soft hands and split pairs), again together with the upcard.

Complicating the ideal strategy even further is the issue of what count(s) to use in guiding play. Usually the card counter selects his counting method to approximate as closely as practical the optimum for betting, and tracks that single count. If he also wants a counting indicator for playing his hand, he has several options. Most ideally, he would simultaneously maintain nine separate and distinct counting registers (in effect, knowing the exact composition of the remaining pack); but this is clearly beyond human capability. Less ideal is to keep a second count, distinct from the bet count, on which to base play decisions. Authorities differ on the single play count that best balances accuracy with simplicity, but generally suggest that it's fairly similar to the bet count. Even more of a compromise is to use just one count for both betting and playing decisions, with the bet count itself being reasonably close to the best possible choice. It can be estimated that the single-count compromise captures roughly half the maximum (i.e., with complete information via nine distinct counts) return increase from play variation; but even the maximum is much smaller than the improvement available from optimal betting. Our analysis here conforms to this compromise.

In the absence of a card count, the best total-dependent playing strategy for a given number of decks – 'best', for the moment, meaning greatest expected return for the first round after a shuffle, with zero true count and pack depth – we'll call "Optimal Basic", or OBS, in an attempt to distinguish it from among the various extant uses of the simpler term 'Basic Strategy'. It's nearly identical to the best composition-dependent strategy for games with four or more decks. In fact, OBS for four decks is also 'best' for any of the larger numbers (six and eight) found in casinos. This particular OBS has come to be called "Generic" (or sometimes "Generic Basic"); it is frequently suggested as an approximation to OBS for games with fewer than four decks (one or two deck games can still be found in some casinos). The expected returns known for these classes of play strategy are assembled in Table 1.

Table 1. Expected return, first hand following a shuffle, from Composition-Dependent, Optimal Basic and Generic Basic play, for various numbers of decks (with rules including dealer stands on soft 17, no doubling or re-splitting of split pairs, double any two-card hand, no surrender)

	Composition-	Total-Dependent	
Decks	Dependent	Optimal	Generic
1	+.000223	-.000147	-.000418
2	-.003489	-.003621	-.003657
4	-.005222	-.005274	-.005274
6	-.005791	-.005819	-.005819
8	-.006074	-.006091	-.006091

But blackjack analysis has been absorbed for decades by the further question: How should the play parameters (whether composition-dependent or total-dependent) be varied to depend on the current true count, and how valid are the simplifications of either truncating the array of variations or ignoring variation altogether? An example of a truncated scheme, the "Illustrious 18", is detailed in Schlesinger (2005); Blackjack Historian (2005) gives a perspective on the I18's origins. Here we revisit these old questions, with some new conclusions.

To begin, we put aside composition-dependent strategies: we consider them difficult to implement in practice and of incremental benefit so modest as to not warrant the effort. We rather focus our analysis on the total-dependent class, and on the single true count optimized for betting. We find that the player needn't learn the complex procedure where each play parameter, independently of the others, has its own variation with true count (frequently called "strategy indices"), a scheme we'll label "Count-Dependent Play". Instead we show that much of the possible advantage a card-counter could gain in this way is achieved by the 'proper' choice of a count-independent strategy. Such a play strategy is easy to use in a casino – just like any basic strategy – and yet improves considerably on OBS. A scheme of this type has been proposed by Marcus (2007) and called Counter Basic Strategy (CBS); we'll adopt his terminology.

Begin by recalling that the expected return varies - and rather strongly - with true count even if the play parameters remain fixed; varying the play parameters improves return by only a modest additional amount. Thus a good strategy approximation is a count-independent one that - importantly - matches the (slightly higher return) count-dependent one at the 'right' choice of count. The most sensible 'right' or 'proper' choice is the one that maximizes the player's average expected cash flow per round (his "yield"), recognizing that he is betting different amounts on different rounds guided by the true count. Since he makes higher bets for higher true counts, the probability distribution of his bet sizes is peaked at a true count that is significantly positive. Play that is optimal at or near that peak - not at zero depth and count - is the 'right' or 'proper' choice for the otherwise count-independent strategy! This is Counter Basic Strategy.

2. Analysis

Our analysis here, and our notation as well, is largely based on our previous work concerning optimal betting, Werthamer (2005) and (2006), which we'll cite as OB-I and OB-II. The player's yield, Y, is defined as his expected cash flow per round, averaged over all rounds between successive shuffles; from OB-I, assuming no risk of ruin,

$$Y = \frac{1}{F}\int_0^F df \int_{-\infty}^{\infty} d\gamma \; B(R)\, R \; p\{\gamma\}\,. \tag{1}$$

Here f is the pack depth, up to a reshuffle penetration F; γ is the true count with the Gaussian probability distribution $p\{\gamma\}$ of equation A3 and OB-I equation (25) ; and $B(R)$ is the optimal bet size, specified later in Results. The quantity $R \equiv \left\langle R_f(\mathbf{d})\right\rangle_\gamma$ is the expectation, for that count and depth, of the round's return over the probabilities d_j of drawing a card of value j, with $j = 1,\ldots,10$.

The contingent expected return expression seems quite formidable at first. But we are able to show that it can be constructed to at least a close approximation from the much simpler (and computationally feasible) form $R_0\left(\langle \mathbf{d}\rangle_\gamma\right)$. This is the return from a hand at zero depth, with the probability of drawing value j on each card of the hand given by the count-dependent expectation

$$\left\langle d_j\right\rangle_\gamma = d_j^0\left(1-\gamma\,\alpha_j/52\right); \tag{2}$$

here $\boldsymbol{\alpha}$ is the counting vector and d_j^0 is the probability of card value j from a freshly shuffled shoe. The rather lengthy derivation of these results, and some additional definitions, are given in the Appendix.

But the expected return also depends on the play parameters: ideally they should be adjusted at each true count and depth so as to maximize the return there, although typically any depth dependence is neglected. We designate such a maximal array of play parameters as $\boldsymbol{\pi}(\gamma)$; and we make this play dependence explicit in the expected return expression, as $R_0\left(\langle\mathbf{d}\rangle_\gamma;\boldsymbol{\pi}(\gamma)\right)$. But this procedure forces parameter changes, or strategy indices, as the count varies. Depending on how the very wide range of possible counts is truncated to drop the more improbable values, the total number of such changes can be well over a hundred (since some parameters change several times over even a truncated count range). The indices are a complicated roster to remember and apply under casino conditions; the Illustrious 18 is just a particular subset of the most influential.

Much easier for the player, of course, is a total-dependent play strategy that has no variation with count. We are free to test the total-dependent play that is optimal at any true count; at zero, equation (2) shows this is just the OBS, with expected return $R_0\left(\mathbf{d}^0;\boldsymbol{\pi}(0)\right)$. In particular, note that although $p\{\gamma\}$ is symmetrical about zero true count, $B(R)$ optimally ramps upward for increasingly positive returns and counts; so the probability distribution of bet sizes, $B(R)\,p\{\gamma\}$, is skewed toward positive counts and shows a peak, under many conditions, or at least a shoulder. Thus we expect that the yield will be maximized for a true count near the peak or shoulder in the bet size distribution. We'll denote the true count that maximizes the yield in this way as γ^*. Then the count-independent play parameters of CBS are $\boldsymbol{\pi}(\gamma^*)$, with resulting maximal yield obtained via integration over $R_0\left(\langle\mathbf{d}\rangle_\gamma;\boldsymbol{\pi}(\gamma^*)\right)$ as per equation (1).

3. Computations

Our computational program comprises a sequence of steps. In outline, the first step writes an algorithmic code (we use Visual Basic as the framework) to compute the expected return from the first hand after a shuffle of a pack with a specific number of decks D. In this step, since the depth at the beginning of the hand is zero, the first card dealt has value j with probability d_j^0 (see Appendix). The play parameters are adjusted to maximize the return at zero count, corresponding to $\boldsymbol{\pi}(0)$. The output, $R_0\left(\mathbf{d}^0;\boldsymbol{\pi}(0)\right)$, successfully reproduces results well known in the literature, such as by Griffin (1999) for one deck and by Manson, et al. (1975) for four decks.

The second step generalizes to a non-vanishing count. The Visual Basic code is extended so that the probabilities of the cards drawn to the hand reflect the true count at its start as per equation (2), giving $R_0\left(\langle\mathbf{d}\rangle_\gamma;\boldsymbol{\pi}(0)\right)$. The "Count-Dependent" play parameters $\boldsymbol{\pi}(\gamma)$ emerge from maximizing the expected return at that true count, and give the count-dependent return $R_0\left(\langle \mathbf{d}\rangle_\gamma;\boldsymbol{\pi}(\gamma)\right)$. As anticipated, these expected returns increase with increasing γ, although with significant curvature away from linear.

The third step fits the computed results for the count-dependent return of the previous step to a low-order polynomial in γ; a quartic is sufficient for accuracy to about 3 significant figures throughout. The five coefficients resulting from the fit are then transcribed into a fourth order truncation of the corresponding Hermite polynomial expansion (see the Appendix) giving an excellent numerical approximation to $\left\langle R_f\left(\mathbf{d};\boldsymbol{\pi}(\gamma)\right)\right\rangle_\gamma$ at non-zero true count and depth. Because of the curvature of $R_0\left(\langle \mathbf{d}\rangle_\gamma;\boldsymbol{\pi}(\gamma)\right)$ with γ, the Hermite terms of order 2-4 convey a dependence on depth.

The fourth step selects several positive trial values for true count γ^*. For each it generates $\boldsymbol{\pi}(\gamma^*)$ from the results of the second step (i.e., maximizing $R_0\left(\langle \mathbf{d}\rangle_{\gamma^*};\boldsymbol{\pi}\right)$ with respect to $\boldsymbol{\pi}$) and then computes $R_0\left(\langle \mathbf{d}\rangle_\gamma;\boldsymbol{\pi}(\gamma^*)\right)$ as a function of γ. A truncated Hermite polynomial expression is generated just as in the third step, approximating the corresponding $\left\langle R_f\left(\mathbf{d};\boldsymbol{\pi}(\gamma^*)\right)\right\rangle_\gamma$.

The fifth step adopts an optimal bet size $B(R)$, specified below; combines it with the expected return approximation from the fourth step; and (having switched to Mathematica as the framework) computes the yield Y, via equation (1), by integrating over γ and averaging over the depth, weighted by $p\{\gamma\}$.

Lastly, the yield values obtained for the several trial values of γ^* are interpolated to arrive at that providing the maximum yield. The corresponding $\boldsymbol{\pi}(\gamma^*)$ then represents our desired CBS. Note, however, that CBS is dependent, at least formally, on the particular choice of bet strategy, as well as on the counting vector and the penetration.

It remains for the future to streamline these computations, here assembled ad-hoc from pieces of earlier work, by consolidating them all into a single Mathematica code.

4. Results

We do not report an exhaustive set of computations for all possible choices of parameters. We rather select a few representative examples to illustrate our methodology and results. We focus on a penetration of $F = 0.8$ (as throughout OB-I and -II); and use the optimal counting vector

$j =$	1	2	3	4	5	6	7	8	9	10
$\alpha_j =$	-1.28	+.82	+.94	+1.21	+1.52	+.98	+.57	-.06	-.42	-1.07

rather than one or another of its more popular (and practical) approximations. This vector is roughly equivalent to Griffin (1999, page 45, last line of Table) and Epstein (1995, page 244, last line of Table 7-12, labeled Thorp Ultimate) but is closer to Epstein (1995, Table 7-11, last line). Further, we display results only for games with a single deck (where yield is the most sensitive to choice of play strategy) and with four decks (representative of six and eight deck games, as well).

We consider the same two betting patterns as in OB-I and -II: the "Weekender" and the "Lifetimer", differing in their risk/reward profile. For each example the bet is a linear ramp, $B(R) = sR/\sigma^2$, where σ^2 is the variance of the return, but capped on the lower end by a positive base bet and on the upper end by a maximum bet (or "spread") of 10 times the base. The coefficients of proportionality (from criteria developed in OB-I) are $s/\sigma^2 = .78$ for the Lifetimer and 2.61 for the Weekender, with 4 decks. These coefficients give the Lifetimer, with a capitalization assumed to be 10^3 base bets, a .13 risk of ruin over 10^6 rounds; the Weekender, with capitalization assumed at only 10^2 base bets, has a .19 risk of ruin in 10^3 rounds. For single deck, the corresponding coefficients are .29 and .43, respectively, and the risk of ruin is negligibly small. The results are shown in Table 2 as the ratio of yield to base bet.

Table 2. Yield ratio for Counter Basic, Optimal Basic and Count-Dependent strategies

Betting style:	Lifetimer		Weekender	
Number of decks:	1	4	1	4
Count-Independent				
Optimal Basic	+.076	+.019	+.006	+.008
Counter Basic	+.091	+.023	+.009	+.011
Count-Dependent	+.083	+.022	+.013	+.011
Effective true count, $\gamma*$	5.1	3.2	4.6	3.5

Clearly, Counter Basic increases the yield over that from Optimal Basic by a meaningful amount, particularly against a single deck.

Table 3 details the Counter Basic Strategy for a single deck, consistent with an effective true count in the vicinity of +5 that is midway between the $\gamma*$ of the Lifetimer and the Weekender; this CBS is, by construction, identical to Count-Dependent play at that true count value. The CBS parameters that differ from those at zero true count (i.e., Optimal Basic) are bolded and italicized.

Table 3. Counter Basic Strategy, single deck (same rules as Table 1)

Upcard	Stand on	Double	Split
A	17; s18	***10,*** 11	A, 8, ***9***
2	***12***; s18	9-11; s17, ***18***	A, 6-9
3	***12***; s18	9-11; s***15***, ***16***, 17, 18, ***19***	A, 2, ***3***, 6-9
4	12; s18	***8***, 9-11; s13-18, ***19***	A, 2, 3, 6-9
5	12; s18	8-11; s13-18, ***19***, ***20***	A, 2, 3, 6-9
6	12; s18	8-11; s13-19	A, 2, 3, 6-9
7	17; s18	***9***, 10, 11	A, 2, 3, 7, 8
8	17; s18	10, 11	A, 8, 9
9	***16***; s19	10, 11	A, 8, 9
10	***15***; s19	***10***, 11	A, 8

Always take insurance

5. Discussion

It is encouraging to compare our CBS result of Table 3 against those of Marcus (2007). Although the respective methodologies are quite different (he uses a simulation program), the resulting plays are quite similar, in particular when interpolating his single-deck charts for spread of 8 with penetrations of .6 and .7, and spread of 12 with penetration .65, to our conditions of spread 10 with penetration .8. Also, Marcus adopts a bet scheme that rises quadratically with true count rather than an optimal linear ramp as here.

Note in Table 2 that the yield ratio for CBS in some cases is actually superior to Count-Dependent play. The comparison is even more dramatic vs. Illustrious 18 play, whose yield is typically around 70% of Count-Dependent. This seeming paradox is understood by recalling that Count-Dependent maximizes the expected return at each true count, but at zero depth; it is not recalculated for each separate depth value. In contrast, CBS maximizes the yield, the dominant contributions to which come from sizable depths, where the true count is more likely to take on large, positive values and the expected returns and bet sizes are correspondingly larger. The depth dependence of the expected return and play strategy, although almost always ignored in the blackjack literature, is a significant influence in our computational results.

It is intriguing to follow this logic still further and note that the optimal counting vector $\boldsymbol{\alpha}$ of the previous section has been defined as maximizing the resulting expected return at zero depth. A better version might be obtained by instead maximizing the yield, with greater influence from conditions of larger depths and higher true counts. Investigation of this generalization, including its quantitative implications, will be pursued separately.

Appendix

We consider a hand that begins after one or more rounds following a shuffle, the previous rounds having used m_j cards of value j, totaling $M = \sum_j m_j$ in number, from a pack of D decks. Define d_j^0 as the probability of drawing value j as the first card following a shuffle: $d_j^0 = 1/13,\ \ j \neq 10;\ \ d_{10}^0 = 4/13$. Then the probability of drawing value j on the first card of the latest round is $d_j = \left(52\,D\,d_j^0 - m_j\right)\big/\left(52\,D - M\right)$. The player is counting with a vector $\boldsymbol{\alpha}$ (assumed balanced, $\sum_j d_j^0 \alpha_j = 0$, and normalized, $\sum_j d_j^0 \alpha_j^2 = 1$) so that the true count at the start of the round is

$$\gamma = \sum_j \alpha_j\, m_j \big/ \left(D - M/52\right) = -52 \sum_j \alpha_j \left(d_j - d_j^0\right). \tag{A1}$$

The probability distribution $\rho\{\mathbf{d}\}$ for the first card drawn to the hand in the absence of counting is given by

$$\rho\{\mathbf{d}\} = \sqrt{2\pi}\ \Delta\ \delta\!\left(\sum_j \left(d_j - d_j^0\right)\right) \prod_j \left\{ \frac{1}{\sqrt{2\pi d_j^0}\,\Delta} \exp\!\left[-\frac{\left(d_j - d_j^0\right)^2}{2\,d_j^0\,\Delta^2} \right] \right\}, \tag{A2}$$

(OB-I equation (24)), where $\Delta \equiv \sqrt{f \big/ 52D\left(1 - f\right)}$ parameterizes the distribution width. Then the distribution contingent on the true count γ becomes

$$\rho\{\mathbf{d} \,|\, \gamma\} = \left[\rho\{\mathbf{d}\} / p\{\gamma\}\right]\ \delta\left(\gamma + 52\boldsymbol{\alpha} \cdot (\mathbf{d} - \mathbf{d}^0)\right), \tag{A3}$$

with normalization $p\{\gamma\}$ given by

$$p\{\gamma\} = \frac{1}{\sqrt{2\pi\tau}} \exp\!\left[-\frac{\gamma^2}{2\tau} \right], \tag{A4}$$

(OB-I equation (25)) and $\tau \equiv \left(52\Delta\right)^2$. The expected card distribution, conditional on γ, becomes $\left\langle d_j \right\rangle_\gamma = \int d\mathbf{d}\ d_j\ \rho\{\mathbf{d} \,|\, \gamma\} = d_j^0 \left(1 - \gamma\,\alpha_j / 52\right)$.

The probability of the κ^{th} card, drawn from a pack depleted to $52D - M$ cards, having value j_κ and conditional on the previous cards being $j_1, \cdots, j_{\kappa-1}$ is

$$d^\kappa\left(j_\kappa; j_1, \cdots, j_{\kappa-1}\right) = \frac{d\left(j_\kappa\right) - \varepsilon_f \sum_{l=1}^{\kappa-1} \delta\left(j_\kappa, j_l\right)}{1 - \varepsilon_f\ (\kappa - 1)}, \tag{A5}$$

where $\varepsilon_f \equiv \left(52D - M\right)^{-1} = \varepsilon_0 + \Delta^2$ increases with depth. Then the return from any hand of K cards can be expressed as a sum of terms, each corresponding to a different configuration of cards drawn and each with the probability factor $\prod_{\kappa=1}^{K} d^\kappa$.

We now have the machinery to develop a closed form expression for the expected return from the hand. To make explicit the dependence of the return on f, and on the specific configuration $\mathbf{d}$ of the current pack, we denote it as $R_f(\mathbf{d})$. Then

$$\left\langle R_f(\mathbf{d})\right\rangle_\gamma = \int d\mathbf{d}\, R_f(\mathbf{d})\rho\{\mathbf{d}\,|\,\gamma\}. \tag{A6}$$

After substitutions of equations (A2) - (A4), of the Taylor series $R_f(\mathbf{d}) = \exp\left(\left(\mathbf{d}-\mathbf{d}^0\right)\cdot\nabla\right)R_f\left(\mathbf{d}^0\right)$, and of the Fourier representation $\delta(x) = \int_{-\infty}^{\infty} d\varphi(1/2\pi)\exp(i\varphi x)$ for the two Dirac delta functions, the resulting multiple integrations can all be performed straightforwardly, so that

$$\left\langle R_f(\mathbf{d})\right\rangle_\gamma = \exp\left\{\frac{1}{2}\Delta^2\left[\hat{\nabla}^2 - \left(\hat{\boldsymbol{\alpha}}\cdot\hat{\nabla}\right)^2\right] - \hat{\gamma}\,\hat{\boldsymbol{\alpha}}\cdot\hat{\nabla}\right\}R_f\left(\mathbf{d}^0\right), \tag{A7}$$

where the ‘caret’ variables help make the notation more compact: $\hat{\alpha}_j \equiv \sqrt{d_j^0}\,\alpha_j$, $\hat{\nabla}_j \equiv \sqrt{d_j^0}\left(\nabla_j - \sum_l d_l^0\nabla_l\right)$, and $\hat{\gamma} \equiv \gamma/52$. Furthermore, applying a Taylor series again on the variable f,

$$\left\langle R_f(\mathbf{d})\right\rangle_\gamma = \exp\left[\Delta^2\left(\frac{1}{2}\hat{\nabla}^2 + \frac{\partial}{\partial\varepsilon_0} - \left(\hat{\boldsymbol{\alpha}}\cdot\hat{\nabla}\right)^2\right) - \hat{\gamma}\,\hat{\boldsymbol{\alpha}}\cdot\hat{\nabla}\right]R_0\left(\mathbf{d}^0\right). \tag{A8}$$

But the sum-of-products form of $R_f(\mathbf{d})$ from the previous paragraph enables the near-cancellation (demonstrated below) so that

$$\Delta^2\left(\frac{1}{2}\hat{\nabla}^2 + \frac{\partial}{\partial\varepsilon_0}\right)R_0\left(\mathbf{d}^0\right) \propto \left(\frac{1}{52D}\right)^2. \tag{A9}$$

Even for a single deck, $(52D)^{-2}$ is negligibly small. As a consequence, to this order of approximation, the expected return, equation (A8), reduces to just

$$\left\langle R_f(\mathbf{d})\right\rangle_\gamma \cong \exp\left[-\frac{1}{2}\Delta^2\left(\hat{\boldsymbol{\alpha}}\cdot\hat{\nabla}\right)^2 - \hat{\gamma}\,\hat{\boldsymbol{\alpha}}\cdot\hat{\nabla}\right]R_0\left(\mathbf{d}^0\right) = \exp\left[-\frac{1}{2}\Delta^2\left(\hat{\boldsymbol{\alpha}}\cdot\hat{\nabla}\right)^2\right]R_0\left(\left\langle\mathbf{d}\right\rangle_\gamma\right). \tag{A10}$$

Equation (A10) establishes that the expected return of every hand in a shoe, in the absence of counting, is very nearly the same as the first following its shuffle. (Some authorities assert that these should actually all be identical; but our derivation does not reveal such an equality.)

We still have to deal with the exponential factor in equation (A10a), to which we apply an expansion in a series of Hermite polynomials, H_n. Thus

$$\exp\left[-\frac{1}{2}\Delta^2\left(\hat{\boldsymbol{\alpha}}\cdot\hat{\nabla}\right)^2 - \hat{\gamma}\,\hat{\boldsymbol{\alpha}}\cdot\hat{\nabla}\right]R_0\left(\mathbf{d}^0\right) = \sum_{n=0}^{\infty}\frac{\Delta^n}{n!}H_n\left(\frac{\hat{\gamma}}{\Delta}\right)\left(-\hat{\boldsymbol{\alpha}}\cdot\hat{\nabla}\right)^n R_0\left(\mathbf{d}^0\right). \tag{A11}$$

We anticipate that truncating the series at a low order provides an adequate approximation for computational purposes. This is born out by computing $R_0\left(\langle \mathbf{d}\rangle_\gamma\right)$ (i.e., equation (A10b) with $\Delta^2 = 0$, corresponding to the first round after a shuffle) as a function of γ and finding that the resulting curve – while not strictly linear as approximated in OB-I and -II - can instead be fit reasonably with a quadratic, and with a quartic to an accuracy much better than 1%. We then insert the 5 coefficients of the fit into a truncation of equation (A11) at order 4; the result now incorporates correctly (and computably) the Δ^2 dependence seen in equation (A10). In other words, the least-squares fit $R_0\left(\langle \mathbf{d}\rangle_\gamma\right) \cong \sum_{n=0}^{4} c_n\, \gamma^n$ (returning to the original variables from the 'caret' ones) translates into the approximation

$$\begin{aligned} \left\langle R_0(\mathbf{d})\right\rangle_\gamma &\cong \sum_{n=0}^{4} c_n \tau^{n/2} H_n(\gamma/\sqrt{\tau}) \\ &= c_0 + c_1\gamma + c_2\left(\gamma^2 - \tau\right) + c_3\left(\gamma^3 - 3\gamma\,\tau\right) + c_4\left(\gamma^4 - 6\gamma^2\tau + 3\tau^2\right). \end{aligned} \tag{A12}$$

To complete our arguments, we need to provide a proof of equation (A9). Begin with a representative product term in the probability of a hand of K cards: substituting equation (A5), such a product term is of the form

$$\prod_{\kappa=1}^{K}\left[\left(d_{j_\kappa}^0 - \varepsilon_0 \sum_{\iota=1}^{\kappa-1}\delta\left(j_\kappa, j_\iota\right)\right)\Big/\left(1-\varepsilon_0(\kappa-1)\right)\right]. \tag{A13}$$

Then apply the operator $\left(\frac{1}{2}\hat{\nabla}^2 + \frac{\partial}{\partial \varepsilon_0}\right)$ to it and carry out the differentiations. Although the manipulation is tedious and the resulting expression is lengthy and cumbersome, rearrangement and careful attention to cancellations among terms shows that it is proportional to $\varepsilon_0 = \left(52D\right)^{-1}$. The co-factor remains complicated but seems not to vanish, either identically or in the limit of $\varepsilon_0 \to 0$; nevertheless, we've arrived at the desired result.

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